Transmission-Line Properties of a Strip Line Between Parallel Planes

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Abstract—The subject is a strip line sandwiched in dielectric between parallel planes (commonly termed "stripline"). In the manner of the author's earlier papers relating to a different type (1964, 1965, 1977), all the significant properties are formulated in explicit form for practical applications. This may mean synthesis and/or analysis. Each formula is a close approximation for all shape ratios, obtained by a gradual transition between theoretical forms for the extremes of narrow and wide strips. The effect of thickness is formulated to a second-order approximation. Then the result is subjected to numerical differentiation for simple evaluation of the magnetic-loss power factor from the skin depth.

The familiar derivation for a thin strip (in terms of elliptic integrals K'/K) is obtained by a simple algorithm of binary stepping with no reliance on tables. This is susceptible of any degree of approximation in closed form and is reversible for synthesis or analysis. It is used to verify a simple empirical formula which is more convenient for differentiation.

In the transition region between the extremes of narrow and wide strips, the effect of thickness is computed by conformal mapping and numerical integration (in place of elliptic integrals). From this reference, a simple empirical formula is verified.

Graphs are given for practical purposes, showing the wave resistance and magnetic loss for a wide range of shape and dielectric. For numerical reading, the formulas are suited for programming on a small digital calculator.

I. Introduction

ONE FORM of strip line is suited for the highest degree of refinement in a printed circuit. It is the familiar symmetrical type made of a printed strip sandwiched in homogeneous dielectric between parallel planes. The symmetry and the planes provide complete shielding. It is distinguished from the strip near a single plane (termed "microstrip") which is half-shielded.

The purpose of this article is to present some improved formulas and graphs similar to those recently published for the strip near one plane [3]. They include not only wave resistance (so-called "characteristic impedance") but also the losses. The effect of strip thickness is simply formulated to enable the evaluation of magnetic loss.

In the vernacular, this type of line is termed "stripline," a term which is not distinctive.

Because the subject line is symmetrical and is imbedded in homogeneous dielectric, its properties can be stated in simpler form. They are presented in formulas and charts which are complete for a strip of any width and moderate thickness.

As in the preceding article [3], the principal component of dissipation is usually the magnetic power factor (PF or $1/Q$). It is strongly dependent on the strip thickness. This loss PF can be evaluated with the aid of the "incremental-inductance rule" published by the author in 1942. Other authors have applied this rule to the subject line [7] but the present approach offers some simplification in computation.

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As before, we shall consider only the lowest mode of wave propagation in the line.

After a list of symbols, the configuration will be defined and the scope of this article will be indicated.

II. Symbols

MKS rationalized units (meters, ohms, etc.)

k	= dielectric constant of sheet of material separating the strip and the ground plane.
R_c	$=377=120\pi=$ wave resistance of a square area of free space or air.
R	= wave resistance of the transmission line formed by the strip between parallel ground planes (of perfect conductor) filled with dielectric (k).
R_1	$=R$ without dielectric ($k=1$).
R_δ	$=R_1$ subject to skin depth (δ) in real conductor.
R/R_1	$=1/\sqrt{k}=$ speed ratio in dielectric (k) relative to free space or air.
r	$=\sqrt{k}\ R/R_c=$ normalized R.
w	= width of strip conductor.
h	= height (separation) of strip from each ground plane.
t	= thickness of strip conductor.
w'	= width of an equivalent thin strip.
Δw	$=w'-w=$ width adjustment for thickness.
δ	= skin depth in the conductor.
p	$=1/Q=$ magnetic power factor (PF) of strip line.
P	$=p\div\delta/h=ph/\delta=$ normalized p.
e	$=2.718=$ base of natural logarithms.
$\exp x$	$=e^x=$ natural exponential function.
$\ln x$	$=\log_e x=$ natural logarithm.
acosh $x=$	anticosh $x=\cosh^{-1}x$.
atanh $x=$	antitanh $x=\tanh^{-1}x$.

III. A Strip Line Between Two Parallel Planes

Fig. 1(a) shows the cross section of the subject line. The "practical" parameters correspond to the preceding article [3]. They are the wave resistance (R) and the dimensions (w,h,t). Here again the thickness is featured, and the

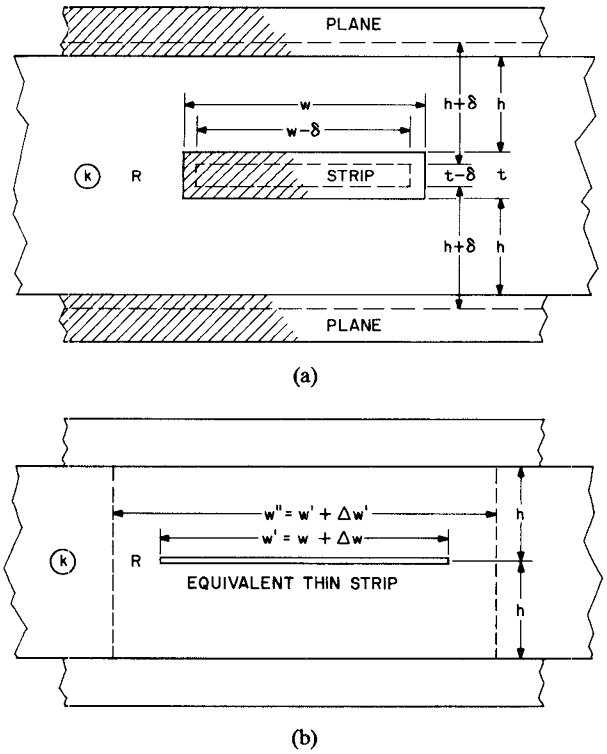

Fig. 1. A strip line between parallel planes. (a) Rectangular cross section. (b) Thin strip of equivalent width.

equivalence between a practical strip and a wider theoretical thin strip (perfect conductor with thickness approaching zero). Fig. 1(b) shows the latter. The equivalence is described in terms of the width adjustment (Δw).

Of particular interest is the well-known wide-strip approximation indicated in Fig. 1(b). The edge effect of a wide thin strip between parallel planes is highly localized so it can be described by one number. This effect is designated by

$$w'' = w' + \Delta w' = w' + \frac{h}{\pi} \ln 16 = w' + 0.88h. \quad (1)$$

This greater width would give the correct wave resistance (R) on the assumption of uniform field within this width and zero outside. The effective width is less to the extent of interaction between the field distortions at the opposite edges. This interaction is small if the width exceeds the height ($w/h > 1$) and will be expressed as another term.

For evaluation of the magnetic PF, the skin effect is indicated in dashed lines. These boundaries are recessed by 1/2 the skin depth ($\delta/2$) so they indicate the actual center of current. The actual boundary is the theoretical current center in a perfect conductor. The change between one and the other is involved in the computation of the magnetic PF. It is assumed that all conductive boundaries are nonmagnetic and have equal conductivity and skin depth.

IV. Scope

As in the preceding article [3], the thrust of this article is to enable explicit synthesis of a line to meet some specifications. The wave resistance (R) is related to the dielectric constant (k) and the shape. On the other hand, the magnetic PF can be decreased by increasing the size, while the shape has a lesser effect. The PF is usually a tolerance rather than a requisite. The wave-speed ratio is taken not to be specified, but is determined by the choice of the dielectric.

Some graphs are introduced here, for reference in various sections. They present the relations needed for the purposes of practical design, and can be read close enough for ordinary purposes. The formulas to be given are intended as an alternative to the graphs, and also to give further insight into the relations. The formulas are designed for programming in a small digital calculator such as the HP-25 or HP-97.

Fig. 2 is a graph of the wave resistance of a thin strip. It is made from the algorithm of binary stepping. The wave-speed ratio (relative to air or free space) for any width ratio is equal to the ratio of wave resistance with and without dielectric (R/R_1).

Fig. 3 is a graph of the thickness effect on the wave resistance without dielectric. It is a small effect with respect to wave resistance but has a greater effect on the magnetic PF. This is generally similar to the first-order effect of thickness as previously stated [3] but is refined and extended to include the second-order effect in some degree.

Fig. 4 is a graph of the normalized magnetic PF ($P = p \div \delta/h$) as evaluated from the thickness effect. The magnetic PF is independent of the dielectric and its normalized value is independent of the size. The thickness parameter (t/h) is chosen as being a property of the laminate, specifically the thickness ratio of the conductive strip and the dielectric sheet.

New formulas are here presented in the main text without derivation. Most of them are empirical formulas providing a gradual transition between narrow and wide extremes. These are first tested against close approximations for overlapping narrow and wide ranges. They are further tested against a set of examples computed by conformal mapping, which cover the transition region between the narrow and wide ranges. Some derivations, not previously available, are given in the Appendixes. Special emphasis is placed on some formulas which are "reversible" in the sense that a formula can be expressed explicitly in a simple form for either analysis or synthesis.

V. A Thin Strip

As in the preceding article [3], simple "reversible" formulas are given for a thin strip of any width. By this is meant that an explicit formula for either analysis or synthesis can be converted to an explicit formula for the other. This conversion is permitted no complication beyond the solution of a quadratic equation. These formulas are empirical in the sense that they must be tested against derived formulas that are exact or at least give a close approximation in some range of shape.

The simple explicit formula is here shown first for synthesis and then for analysis. The normalized resistance

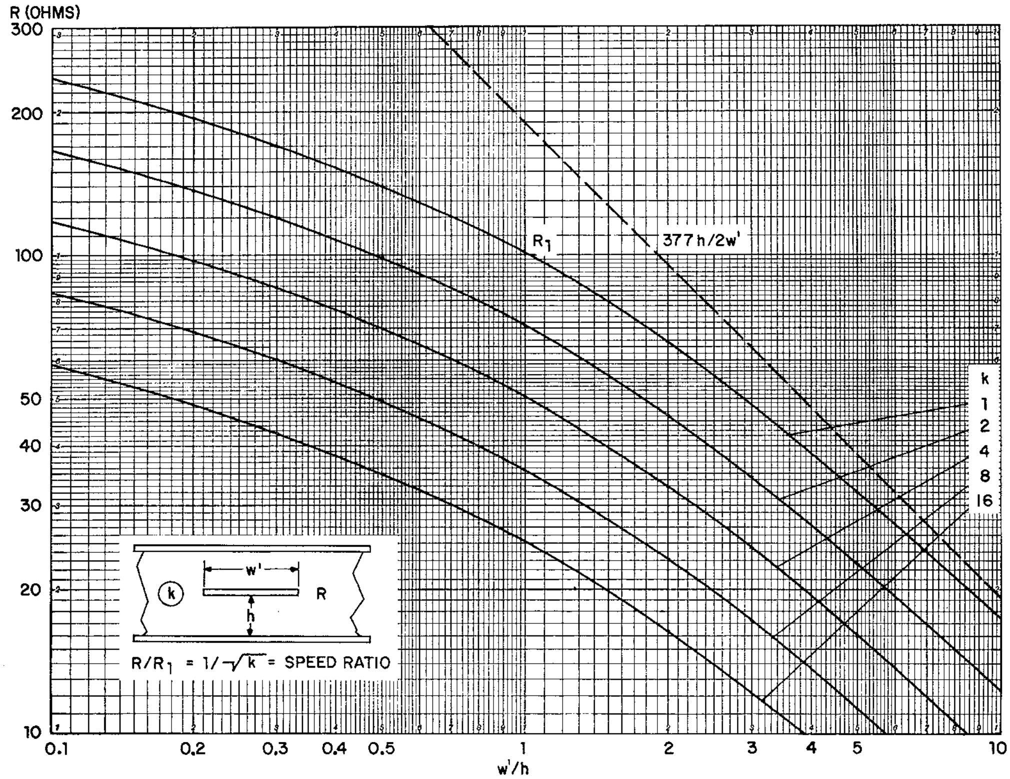

Fig. 2. The wave resistance of a thin strip in dielectric between parallel planes.

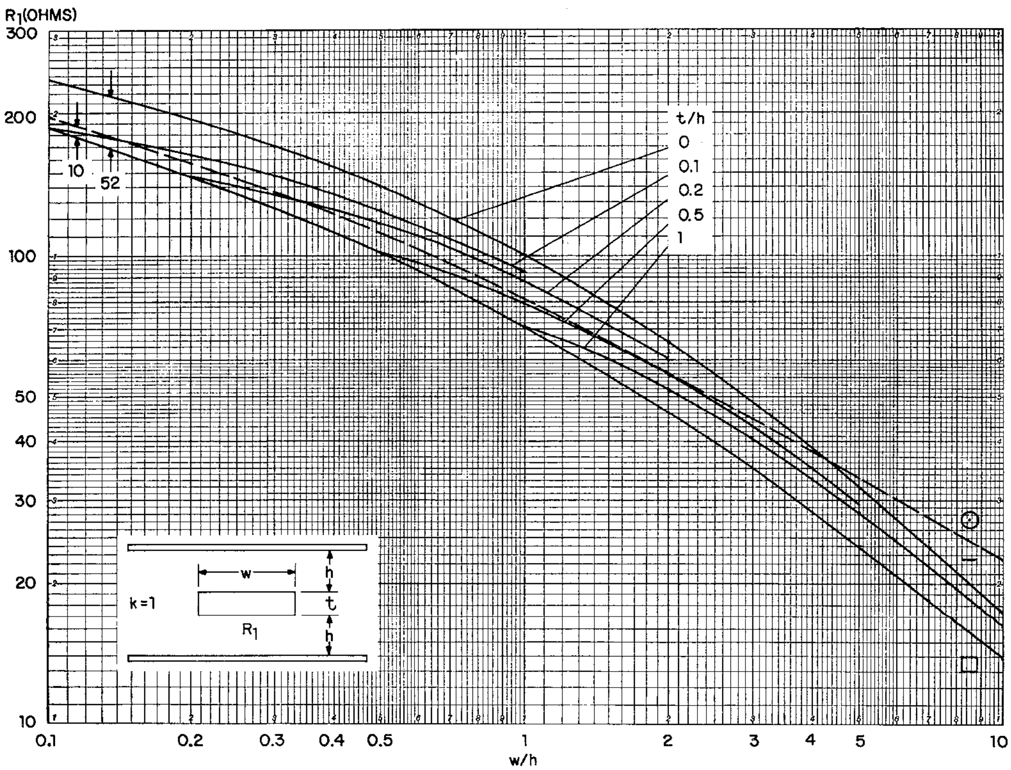

Fig. 3. The wave resistance of a strip without dielectric, showing the effect of thickness.

$$r=\sqrt{k}\ R/R_c=\sqrt{k}\ R/377;\quad R=rR_c/\sqrt{k}\ =377r/\sqrt{k} \tag{2}$$

is introduced for convenience.

$$w'/h=\frac{16}{\pi}\,\frac{\sqrt{(\exp 4\pi r-1)+1.568}}{(\exp 4\pi r-1)} \tag{3}$$

$$R=\frac{30}{\sqrt{k}}\ \ln\left\{1+\frac{1}{2}(16h/\pi w')\Big[(16h/\pi w') + \sqrt{(16h/\pi w')^2+6.27}\ \Big]\right\}. \tag{4}$$

The relative error is < 0.005 of R for $w'/h<20$.

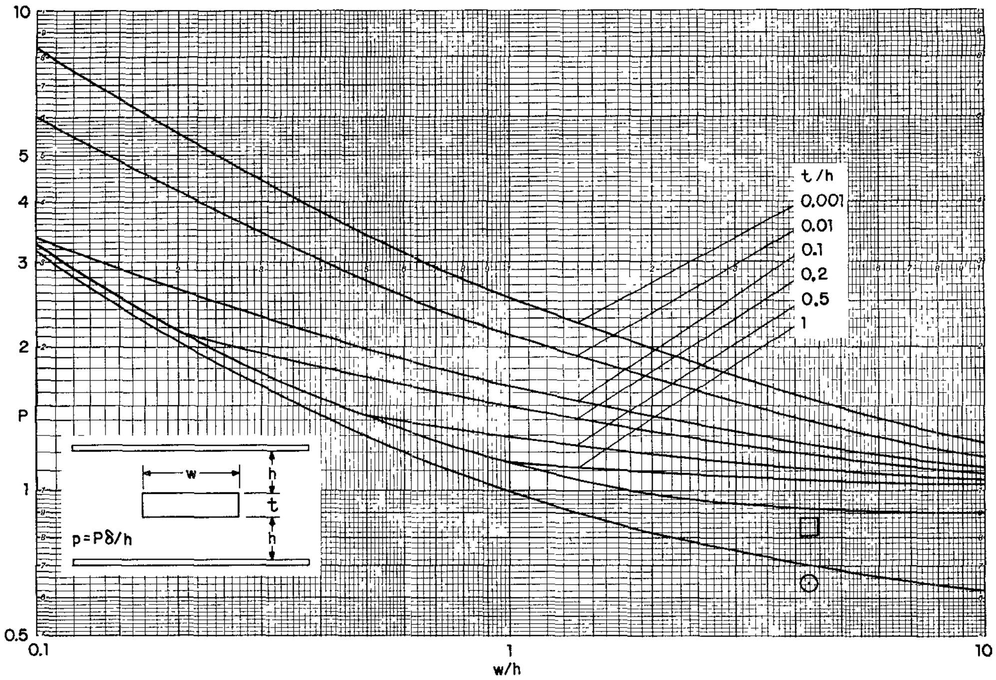

Fig. 4. The magnetic power factor of a strip, showing the effect of thickness.

This formula is based on the "narrow" approximation. The first term is retained. The second term is retained in form and approximately in amount. The latter is based on the two-wire approximation for a thin strip, as described in [3]. However, the form is arranged to approximate also the first term of the "wide" extreme, in respect to its slope and its zero for infinite width. The second term of the "wide" is retained in form but not in amount. The two constants are a slight departure from the following theoretical values for the "wide" extreme, to give closer approximation over the range of most interest:

$$1.568 \text{ instead of } 1.522=(\pi^2/8)^2$$
$$6.27 \text{ instead of } 6.088=(\pi^2/4)^2.$$

From another viewpoint, the two constants are a slight departure from the following theoretical second terms for the "narrow" extreme:

$$1.568 \text{ instead of } 1.667=5/3$$
$$6.27 \text{ instead of } 6.667=20/3.$$

It is notable that the above forms are close to a simple (reversible) formula with these theoretical constants and the following (rather close) bounds for narrow and wide extremes:

$$\frac{16}{\pi}\,\frac{\exp 2\pi r}{\exp 4\pi r-1}=\frac{8/\pi}{\sinh 2\pi r}<w'/h<\frac{\pi}{\sinh 2\pi r}. \quad (5)$$

This approximate formula is tested against the exact formula which is derived by conformal mapping. The latter is related to the ratio K'/K of elliptic integrals, and can be evaluated to any degree of approximation by the method of binary stepping, described in Appendix A.

In Appendix B, there is given the asymptotic formula for "wide" strips of any thickness. It is a close approximation if $w/h>1$. For wide strips (especially if the thickness is small) it is closer than the numerical integration with a moderate number of intervals.

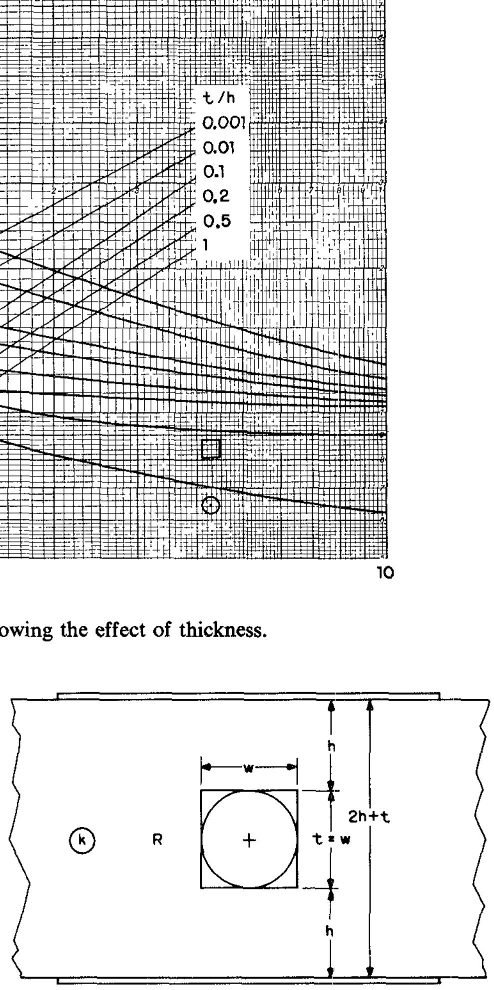

Fig. 5. A wire of square or inscribed circular cross section between parallel planes.

VI. Square or Circular Cross Section

As an extreme departure from a thin strip, a square or circular cross section is considered. Some peculiarities of these shapes are presented in the preceding article [3]. Fig. 5 shows the dimensions between parallel planes, with equal width and thickness ($w=t$). As in the case of the rectangular cross section, the "height" (h) is the separation of the inner conductor from either plane, so the spacing between the planes is $2h+t$. This convention is favored, because the separation places a lower bound on the magnetic PF.

Here there is no exact formula for all shape ratios. There is a "narrow" formula for square or circular, but that does not give the peculiar variation of the PF with shape. Appendix B gives a derivation for a "wide" square cross section. The author's 1955 article [9] gives a derivation for a circular cross section over the entire range of shape ratio.

For the square cross section, a simple "reversible" formula has not been found for a close approximation over the entire range of shape ratio. As will be seen, the more general rectangular cross section yields an explicit formula for analysis, which includes the square. That is what is needed for evaluating the magnetic PF. Synthesis for the square still requires computation by trial or reading from a graph such as Fig. 3.

For the circular cross section, the following "reversible" formula has been designed and tested.

$$w/h=\frac{2}{\left[1+\dfrac{(\exp 8\pi r-1)^2}{2.628\,(\exp 8\pi r-1)+2}\right]^{1/4}-1} \tag{6}$$

$$R=\frac{15}{\sqrt{k}}\ln\left[1+1.314g+\sqrt{(1.314g)^2+2g}\,\right] \tag{7}$$

in which

$$g=(1+2h/w)^4-1$$

$$(4/\pi)^4=2.628=2(1.314).$$

The relative error is < 0.005 of R.

For each extreme of "narrow" and "wide," this formula matches the first term, while the second term is retained in form and approximately in amount. The result is a remarkably close approximation in the transition between extremes.

If the diameter occupies less than one-half the space between the planes, the following "narrow" formulas are simpler and very close (relative error is < 0.005 of R).

Square:

$$w/h=\frac{1}{0.4634\exp 2\pi r-0.5} \tag{8}$$

$$R=\frac{60}{\sqrt{k}}\ln\frac{4/\pi}{1.18}(1+2h/w)=\frac{60}{\sqrt{k}}\ln 1.079(1+2h/w). \tag{9}$$

Circular:

$$w/h=\frac{1}{(\pi/8)\exp 2\pi r-1/2} \tag{10}$$

$$R=60\ln\frac{4}{\pi}(1+2h/w)=60\ln(1+2h/w)+14.49. \tag{11}$$

VII. Strip Thickness and the Loss Power Factor

The approach here is the same as in the preceding article [3]. Appendix B herein gives the effect of any thickness for a "wide" strip between parallel planes. The effect for a "narrow" strip is the same for one or two planes.

The present strip between planes is susceptible of simpler evaluation by conformal mapping and numerical integration, as appears in Appendix C. Therefore, a set of numerical examples has been computed to cover a wide range of shape ratios, especially in the transition region between narrow and wide extremes. From these examples, the previous formulas for the effect of thickness have been refined to give closer approximation for moderate thickness.

Here one should note the basis for the width adjustment, as shown in Fig. 1. The strip thickness is associated with greater separation of the parallel planes in order to keep the same "height" between the strip and either plane. This relation is reflected in the exponent (m) of the "narrow" term in the formulas to be given. Also the "wide" term is modified to give closer approximation for greater thickness.

Each formula for width adjustment is presented here, as before, in terms of the actual width (w) or the equivalent-thin-strip width ($w'=w+\Delta w$).

$$\frac{\Delta w}{t}=\frac{1}{\pi}\ln\frac{e}{\sqrt{\left[\dfrac{1}{4h/t+1}\right]^2+\left[\dfrac{1/4\pi}{w/t+1.10}\right]^m}} \tag{12}$$

or

$$\frac{1}{\pi}\ln\frac{e}{\sqrt{\left[\dfrac{1}{4h/t+1}\right]^2+\left[\dfrac{1/4\pi}{w'/t-0.26}\right]^m}} \tag{13}$$

in which

$$m=\frac{2}{1+t/3h}=\frac{6}{3+t/h}.$$

The relative error is < 0.015.

This adjustment enables a width conversion either way between strips with or without thickness. Its small relative error makes a much smaller contribution to the relative error of effective width and R. It does relate directly to the magnetic PF in view of the dependence of losses on the thickness. As before, the form of this formula gives asymptotic approximation for narrow and wide extremes, and a remarkably close approximation through the transition therebetween.

For loss computation, the actual width and thickness (w,t) are converted to the width of an equivalent thin strip ($w'=w+\Delta w$). Then the thin-strip formula (R_1 of w') can be used for differentiation with respect to the actual dimensions (w,h,t).

As indicated in Fig. 1, each dimension is incremented by $\pm\delta$ and the same formula is used again to obtain R_δ. Then the (small) loss PF is computed by the incremental-inductance rule:

$$p=\frac{R_\delta-R_1}{R_\delta}=1-R_1/R_\delta=\ln R_\delta/R_1\ll 1. \tag{14}$$

Here again, a normalized form for loss PF is used, which gives the effect of shape, independent of the size, frequency, and conductor material. It is normalized to the height (h).

$$P=p\div(\delta/h)=p(h/\delta);\quad p=P(\delta/h). \tag{15}$$

The reference (δ/h) is the nominal PF of a very wide strip.

In computing the normalized PF (P), the value of skin depth is immaterial if it is sufficiently small to approach the limiting behavior of the skin effect (which is usually present). Also it must not approach the sensitivity of the computer. In a computer giving 9 decimal places, a fair compromise is: $\delta/h = 0.0001$. Then the skin effect is well represented if all dimension ratios exceed 0.001.

For evaluation of a resonator made of strip line, the loss PF (or dissipation factor or $1/Q$) is usually the most significant factor. The wave R is incidentally relevant in the circuit application of the resonator. The loss PF of the magnetic field is evaluated by the simplest formulas (R_1 and Δw without dielectric). For any shape, the value of P enables a computation of the size of the cross section to realize a value of p:

$$h = P\delta/p = P\delta Q. \tag{16}$$

The graphs in Fig. 4 show the loss PF in terms of P for a wide range of shapes. The common reference is the height (h) and the thickness ratio (t/h) because they may be fixed by a dielectric sheet and a conductive sheet bonded thereto.

Referring to Fig. 4, the comments in the preceding article [3] are generally still applicable. Here are some comparisons of the present line with that strip near one plane. The magnetic loss PF is greater for the same specified dimensions because there is less space occupied by the field around the strip. Different comments are applicable to the opposite extremes of width.

A "narrow" strip has the same resistance by [8] but lesser reactance as measured by the reduction of wave resistance in free space by this amount:

$$60 \ln \pi/2 = 27.09\ \Omega. \tag{17}$$

A "wide" strip approaches 1/2 the resistance and reactance, but there is less volume occupied by the field beyond either edge of the strip. This reduces the ratio of reactance over resistance so the PF is increased, as appears on Fig. 4. Typically the increase is around 1/5. For a "wide" strip, the same levels are approached.

The greater PF, with the complete shielding provided by the two planes, may not impose a net disadvantage. The space above the upper plane can be utilized without restriction. Without the upper plane, more space above the strip must be kept clear to avoid further losses in the nearby field. One component of such losses is the radiation that is inherent with incomplete shielding.

The previous article gives procedures for computation for analysis or synthesis. The present model is much simpler because the effective dielectric constant is that of the material, which is specified and determines the speed ratio.

a) For synthesis from graphs:
   (i) specify R and k, h, t; $R_1 = R\sqrt{k}$;
   (ii) read w/h from R_1 on Fig. 3;
   (iii) state skin depth;
   (iv) read P on Fig. 4; $p = P\delta/h$.

b) For synthesis from formulas:
   (i) specify R and k, h, t; $R_1 = R\sqrt{k}$;
   (ii) compute w'/h by (3);
   (iii) compute $\Delta w/t$ by (13); $w = w' - \Delta w$.

c) For analysis from graphs:
   (i) specify k, w, h, t;
   (ii) read R_1 on Fig. 3: $R = R_1/\sqrt{k}$;
   (iii) (iv) same as a).

d) For analysis from formulas:
   (i) specify k, w, h, t;
   (ii) compute $\Delta w/t$ by (12); $w' = w + \Delta w$;
   (iii) compute R by (4);
   (iv) state skin depth;
   (v) compute PF by (14).

The PF computed as above, from formulas (4), (12) for all shape ratios, has been checked against the "wide" formula (44), which is much closer if $w/h > 1$. For all thicknesses up to the square cross section, it is concluded that the above (d) comes within 0.05 of PF, or within 0.015 of PF if $t/h < 0.5$.

VIII. Conclusion

The transmission-line properties of a strip between parallel planes are evaluated in simple formulas, each one adapted for all shape ratios. The presence of a homogeneous dielectric filling has a simple effect on the wave resistance and speed ratio but no effect on the magnetic PF which is usually the principal component of losses. The latter is stated from the viewpoint of analysis, which is usually what is needed.

The advance over previous publications appears mainly in two areas:

a) a relation is expressed explicitly by a single simple formula for the entire range of shape ratio;

b) the width adjustment for thickness is formulated and used for evaluation of magnetic loss.

Each formula is an empirical relation obtained by designing a gradual transition between known simple formulas for both extremes of narrow and wide shapes.

All formulas are designed for ease of programming on a small calculator such as the HP-25 or HP-97. Particularly, the digital calculator enables the numerical differention (for loss evaluation) which is here used to realize a great simplification. While beyond the scope of this article, the writer would welcome inquiries relating to the programs for the HP-25 or HP-97, some of which may be available on request.

In general, the subject line presents problems of evaluation which are simpler than those of a strip near one plane. Some of the approaches that were developed for the other form are found interesting and helpful for the subject line, however simpler.

Appendix A
Exact Computation of a Thin Strip by Binary Stepping

For the extreme of either a narrow strip or a wide strip, a simple formula is known which converges to any degree of approximation. A simple iterative algorithm of exact transformations is available for analysis or synthesis of any shape ratio in the intermediate range. The algorithm involves binary stepping of the wave resistance and a corresponding (nonbinary) stepping of the strip width. Specifically, one inward step gives the second term of a converging series, and each further step gives another term. The transformation for each step is made by the same simple formula, which is reversible for stepping in either sense, and is applicable to analysis or synthesis.

The binary stepping will be derived here by a simple pair of inversions in conformal mapping. The stepping is accomplished without reference to the actual relation between the shape ratio and the wave resistance. Then the latter is introduced to establish one end of the "staircase."

Fig. 6 shows a pair of cases differing by a binary step of wave resistance (R). Fig. 7 shows the pair of inversions by which the corresponding step of shape ratio (w/h) is derived. The latter will appear in "reversible" form adaptable for analysis or synthesis.

The symbols used in this exercise are different in these respects:

R', R'' = the (lesser, greater) values of wave resistance ($2R' = R''$).
w', w'' = the corresponding (greater, lesser) values of strip width ($w' > 2w''$).
h = π = reference height.
θ = half-angle of gap in pair of arcs of a circle.

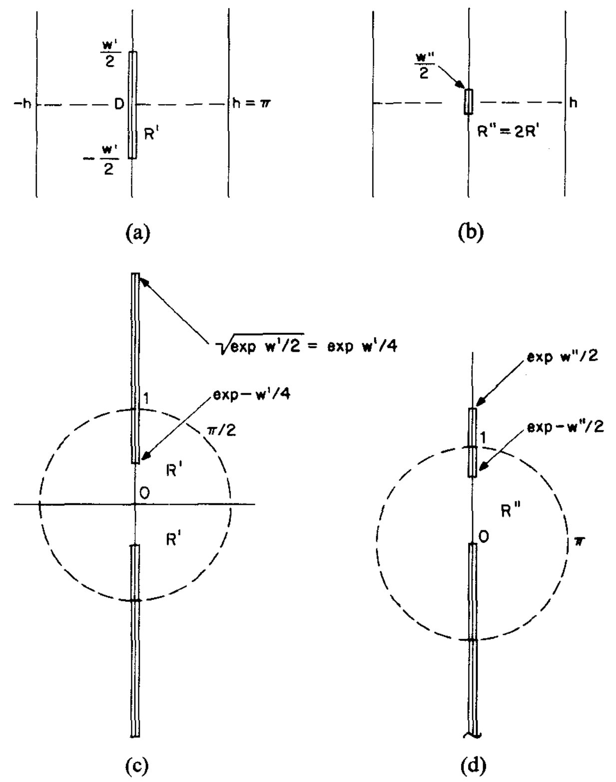

Fig. 6. Description of strip lines of related widths.

Fig. 6(a) shows the cross section of any one width (w'/h) of a thin strip between parallel planes. Fig. 6(b) shows the strip of the lesser width (w''/h) to be derived. The wave resistance is doubled ($R' \times 2 = R''$).

Fig. 6(c) shows the mapping of one strip onto the upper and lower half-planes of a balanced strip line which therefore has double the resistance. The mapping function is $\pm\sqrt{\exp}$ as indicated. The lines of the parallel planes are folded out to form a single line and the strip is imaged to form a balanced pair.

Fig. 6(d) shows the mapping of the other strip and its shields onto a vertical line by the exponential function as indicated. The shield becomes the lower half of an infinite line. The wave resistance is unchanged.

Fig. 7 shows the inversion of the strips of different widths onto the same arcs of a circle, thereby establishing the relations for $2R' = R''$. The arcs are symmetrical and are separated by an angle (2θ) on the circle.

Comparing Figs. 6(c) and 7(a):

$$\tan\theta = \exp - w'/4.$$

Then comparing Figs. 6(d) and 7(b):

$$\frac{\exp w''/2}{\exp - w''/2} = \frac{1/\tan 2\theta + \tan 2\theta}{\tan 2\theta}$$

$$\exp w'' = (1/\sin 2\theta)^2$$

$$(\sin 2\theta)^2 = \exp - w''; \quad (\cos 2\theta)^2 = 1 - \exp - w''$$

$$(\tan 2\theta)^2 = \frac{1}{\exp w'' - 1} = \left(\frac{2\exp - w'/4}{1 - \exp - w'/2}\right)^2$$

$$= (1/\sinh w'/4)^2$$

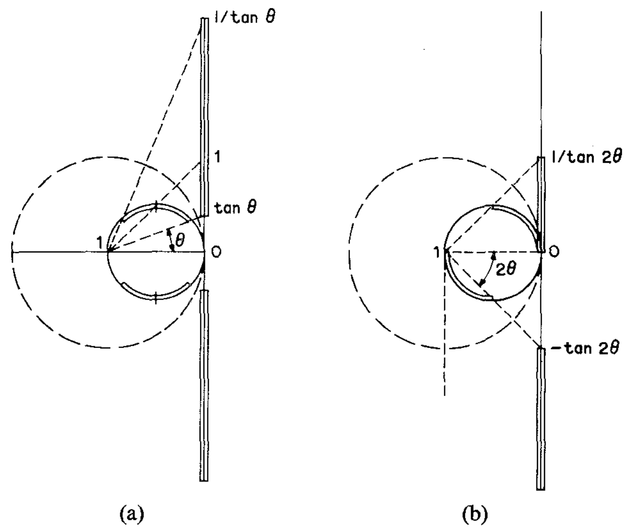

Fig. 7. The circular arcs related to balanced and unbalanced strip lines by inversion.

$$\exp w'' = 1 + (\sinh w'/4)^2 = (\cosh w'/4)^2$$

$$\exp w''/2 = \cosh w'/4$$

$$w'' = 2 \ln \cosh w'/4$$

$$\frac{w''}{h} = \frac{2}{\pi} \ln \cosh \frac{\pi}{4}\frac{w'}{h} \tag{21}$$

$$= \frac{1}{2}\frac{w'}{h} - \frac{2}{\pi} \ln \frac{2}{1 + \exp - \frac{\pi}{2}\frac{w'}{h}}. \tag{22}$$

This is the transformation for double resistance. The second form is preferred for understanding and computation. The first term is 1/2 the width, which is the principal term for wide strips. The two terms have a small difference for narrow strips, giving the asymptotic form:

$$\frac{w''}{h}=\pi\left(\frac{w'}{4h}\right)^2 \ll 1. \qquad (23)$$

One-half the resistance can be obtained by the reverse transformation, which is easily stated from (21):

$$\frac{w'}{h}=\frac{4}{\pi}\ \text{acosh exp}\ \frac{\pi}{2}\frac{w''}{h} \qquad (24)$$

$$=2\frac{w''}{h}+\frac{4}{\pi}\ln\left(1+\sqrt{1-\exp-\pi\frac{w''}{h}}\right). \qquad (25)$$

The asymptotic form for narrow strips is:

$$\frac{w'}{h}=4\sqrt{\frac{w''}{\pi h}} \ll 1. \qquad (26)$$

The complete form for the direct or reverse transformation is exact so it survives the transition region in either direction between the extremes of wide and narrow strip.

As a matter of passing interest, the shapes diagramed ($\theta=1/16$ circle) give:

$R'=377/4\sqrt{k}$ and $R''=377/2\sqrt{k}$ in a dielectric (k).

The second term from the wide extreme is particularly interesting because it gives the interaction between the opposite edges of the strip. This effect becomes apparent if (22) is rearranged as follows:

$$\frac{1}{2}\left(\frac{w'}{h}+\frac{1}{\pi}\ln 16\right)=\left(\frac{w''}{h}+\frac{1}{\pi}\ln 16\right)-\frac{1}{2\pi}\exp-\pi w''/h\pm\cdots. \qquad (27)$$

The last term shown is the second term of the series, the departure from 1/2 the effective width indicated in Fig. 1(b). This is the basis for the interaction mentioned in Appendix B.

The second term from the narrow extreme is interesting, especially for comparison with just one plane [3]. The first approximation for the extreme is:

$$r''=\frac{1}{2\pi}\ln\frac{16h}{\pi w''}\quad \frac{\pi w''}{16h}=\exp-2\pi r''. \qquad (28)$$

One step inward ($r'=r''/2$) gives the second approximation:

$$\frac{w'}{h}=\frac{4}{\pi}\ \text{acosh exp}\ \frac{\pi}{2}\frac{w''}{h}=\frac{4}{\pi}\sqrt{\pi\frac{w''}{h}}\left(1+\frac{\pi}{12}\frac{w''}{h}+\cdots\right)$$

$$=\frac{16}{\pi}(\exp 2\pi r')\left(1+\frac{4}{3}\exp-4\pi r'+\cdots\right)$$

$$R'=\frac{60}{\sqrt{k}}\left[\ln\frac{16h}{\pi w'}-\frac{\pi^2}{192}\left(\frac{w'}{h}\right)^2-\cdots\right]. \qquad (30)$$

In free space ($k=1$) the second term is $(w'/h)^2\times 3.1\ \Omega$. In comparison with just one plane, this term is greater (in the ratio $\pi^2/6=1.64$) and its associated wave resistance is smaller (by 27 Ω). The greater second term could have been evaluated by summing the series of images of the pair of small wires that is equivalent to a thin strip [3]. Note that $\pi^2/6=\zeta(2)=\Sigma 1/n^2$.

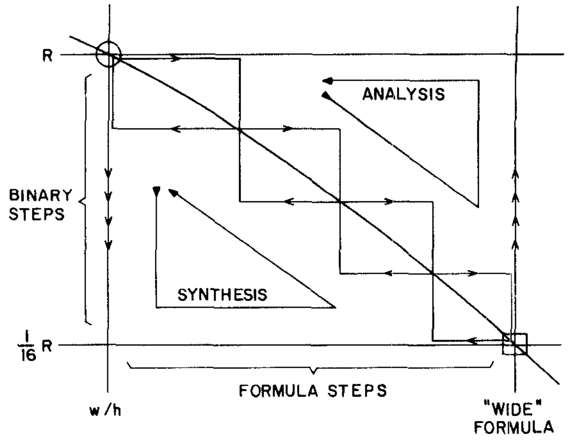

Fig. 8. Flow chart of binary stepping.

For either analysis or synthesis, a practical explicit algorithm may be based on a prescribed number of binary steps from either extreme. The one here proposed is based on 4 steps inward from the wide extreme, for these reasons:

a) the "wide" formula is simpler (free of ln or exp functions of the shape ratio);
b) there is a practical lower bound of the width dimension, below which there is no interest in very close approximation for a thin strip;
c) the set of 4 steps covers the range of the graphs ($w/h>0.1$) with very small relative error ($<10^{-6}$ of R) for use in checking approximate formulas.

The "wide" formula is noted here for use in this algorithm:

$$w'/h=1/2r'-\frac{1}{\pi}\ln 16=377/2R'\sqrt{k}-\frac{1}{\pi}\ln 16 \qquad (31)$$

$$R'=\frac{377/2\sqrt{k}}{w'/h+\frac{1}{\pi}\ln 16} \qquad (32)$$

in which

$$\frac{1}{\pi}\ln 16=0.882\,542.$$

Here the wave resistance of free space is taken to be 377 Ω as a reference for very close approximation.

Fig. 8 is a flow chart of the following procedures for analysis and synthesis. The coordinates are those of Figs. 2 and 3.

For analysis, use this procedure:

a) step up the width ratio w/h by (25) 4 times to get w'/h;
b) compute $R'=R/16$ by (32);
c) multiply by 16 to get R.

For synthesis, use this procedure:

a) divide R by 16 to get R';
b) from R', compute w'/h by (31);
c) step down the width by (22) 4 times to get w/h.

Binary stepping is inherently a special case of the Gauss transformation in the iterative evaluation of an elliptical integral by stepping from either extreme. In general, unequal steps are involved. Binary stepping is a peculiarity of the ratio K'/K, which is related to the subject configuration. Binary stepping is seldom described [11]. Its derivation by inversions is peculiar to this configuration.

Appendix B
Wide Rectangular Cross Section

A wide strip between parallel planes has its two edges isolated to such a degree that their local effects have little interaction. This offers a major simplification relative to a strip near a single plane. A useful approximation is obtained from an exact analysis of a very wide strip, ignoring interaction between the edges.

Referring to one edge of the strip in Fig. 1(a), a conventional analysis by conformal mapping can yield the edge adjustment for an equivalent thin strip as in Fig. 1(b). Equivalence is based on equal capacitance, inductance and wave resistance (R) of perfect conductors and the same homogeneous dielectric (k). The thickness (t) of the actual width (w) is replaced by an extra width (Δw) of the equivalent thin strip ($w' = w + \Delta w$).

The height (h) is the separation between the strip and either plane. It is held the same for two reasons:

a) in practice, it may be the thickness of a dielectric sheet;
b) in theory, it imposes a fundamental limitation on the reduction of the magnetic loss PF.

This rule may cause some confusion in the difference of spacing between the parallel planes in the different equivalent forms. Some such confusion cannot be avoided.

The edge adjustment in the limiting case of a "wide" strip is formulated as follows [10]:

$$\frac{\Delta w}{t} = \frac{1}{\pi}\left[\ln\left(1+\frac{4h}{t}\right) + \frac{4h}{t}\ln\left(1+\frac{t}{4h}\right)\right]. \tag{41}$$

In the limit of a thin strip ($t/h \to 0$):

$$\frac{\Delta w}{t} = \frac{1}{\pi}\left[\ln\left(1+\frac{4h}{t}\right)+1\right] = \frac{1}{\pi}\ln e\left(1+\frac{4h}{t}\right). \tag{42}$$

This is the "wide" term in formula (12), (13). It is an approximation for moderate thickness ($t/h<1$).

The isolation between the two edges is measured by the ratio $\exp -\pi w/h$ in the space between the strip and either plane. The interaction is proportional to this ratio and the factors of field distortion at the respective edges.

For a thin strip, the first-order interaction decreases the effective width by the amount

$$\frac{h}{2\pi}\exp -\pi w/h. \tag{43}$$

In the intermediate shape ($w=h$), this amount is 0.007 h, so the relative effect is a small fraction (about 0.004 of R).

Thickness decreases the distortion at the edges, and also increases the effective width, so the relative effect is still less.

Taking into account the complete width adjustment (41) and the first-order interaction (27), a very close approximation for a "wide" strip is obtained.

$$R = \frac{188.5}{\sqrt{k}} \cdot \frac{1}{\dfrac{w}{h} + \dfrac{4}{\pi}\left[\ln 2\left(1+\dfrac{t}{4h}\right) + \dfrac{t}{4h}\ln\left(1+\dfrac{4h}{t}\right) - \dfrac{1}{8}\exp -\pi\dfrac{w}{h}\right]}. \tag{44}$$

The relative error is < 0.002 if $w/h>1$, and increases slowly for lesser width. This formula for analysis for a "wide" strip is closest for comparison with (4) and (12) which cover the entire range of shape ratio. In particular, it is closest for computation of P, which is sensitive to the derivative.

The square cross section is a special case susceptive of analysis (but not of explicit synthesis) from formula (41) for the width adjustment. The interaction of the edges is small enough that a close approximation is obtained (relative error < 0.002 of R) for any width exceeding that shown in Fig. 5 ($w/h=1$). For any lesser width, compute the circular cross section and then apply the applicable one of these rules:

a) divide the width by 1.18 or
b) increase R by 10 Ω.

Appendix C
Exact Computation by Conformal Mapping

As in the preceding article [3] a complete computation by conformal mapping offers a direct evaluation of some examples without restriction as to shape of cross section ($w/h, t/h$). It is needed for the effect of thickness (in terms of t/h and/or t/w). The examples are useful for verification of approximate formulas, especially in the range of transition between the extremes which are closely approximated by simple formulas. Here again, the algorithm is numerical integration of the space gradient. It is elementary in contrast with elliptic integrals which might be adapted to this configuration. It does not offer an explicit formulation. Here, however, the primary consideration is the effect of thickness on the width of an equivalent thin strip (for equal R).

Fig. 9(a) shows the contour in space, with identification of the singular points in the upper half-plane (on coordinates $z=x+jy$). Fig. 9(b) is a graph of the space gradient on the boundaries as mapped on a straight line (coordinate u). The mapping has quadrantal symmetry, one quadrant being included in the graph. All space around the strip between planes (a) is mapped on all space around the coplanar strip in a gap in one plane (b).

The space gradient is formulated by inspection, as follows:

$$z' = |\partial z/\partial u| = \left|\frac{2}{1-u^2}\right| \sqrt{\left|\frac{1-u_2^2}{1-u_1^2}\right| \left|\frac{u^2-u_1^2}{u^2-u_2^2}\right|}. \tag{50}$$

Only the area ratios are significant, so the scales can be arbitrarily chosen.

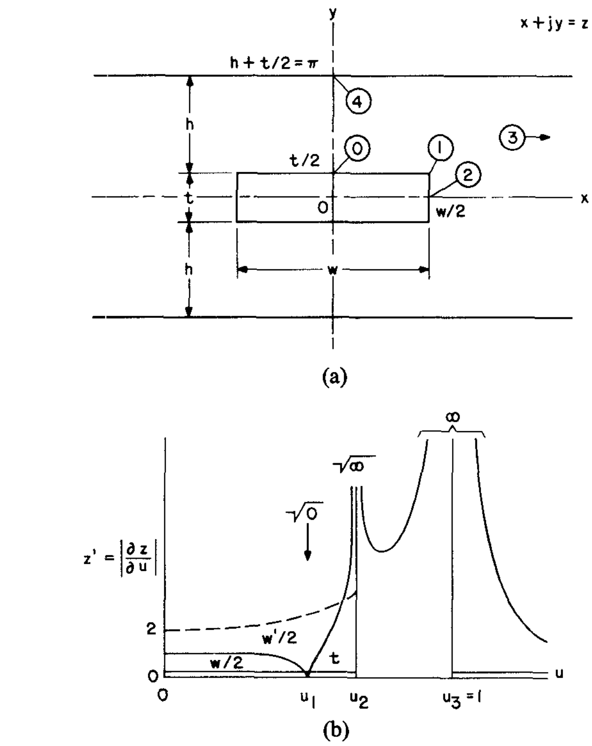

Fig. 9. Conformal mapping of the cross section of the strip line. (a) Contour in space. (b) Space Q radient. (Each area = one dimension on the contour in space.)

The constant coefficients are chosen to give a unit pole (∞) at $u=\pm 1$. This translates to a step of $j\pi$ in integration, which makes $h+t/2=\pi$ on the space contour. Therefore:

$$w/\pi \text{ represents } \frac{w}{h+t/2}=\frac{2}{2h/w+t/w}$$

$$t/\pi \text{ represents } \frac{t}{h+t/2}=\frac{1}{h/t+1/2}$$

$$h/\pi \text{ represents } \frac{h}{h+t/2}=\frac{2}{2+t/h}.$$

In the center (at $u=0$):

$$z'_0=2\frac{u_1}{u_2}\sqrt{\frac{1-u_2^2}{1-u_1^2}}<2\frac{u_1}{u_2}<2. \quad (51)$$

For comparison with a thin strip ($t\to 0$; $u_1\to u_2$):

$$z'=\left|\frac{2}{1-u^2}\right| \quad z'_0=2. \quad (52)$$

This is susceptible of simple analytic integration:

$$\frac{\pi}{2}w'/h=w'/2=\ln\frac{1+u_2}{1-u_2}=2\ \text{atanh}\ u_2 \quad (53)$$

$$u_2=\tanh w'/4=\frac{1-\exp -w'/2}{1+\exp -w'/2}=1-\frac{2}{1+\exp\frac{\pi}{2}w'/h}. \quad (54)$$

This is an explicit reversible formula relating the thin-strip width ratio (w'/h) to the coplanar-strip width ratio (u_2).

The numerical integration of the area (t) is complicated by the half-pole ($\sqrt{\infty}$) at one bound. Some of the common rules fail because they include the end point. Taking the midpoint of each interval, as shown in Fig. 10, this defect is avoided, but a large error remains in the area of the last interval. The relative error of the sum is of the order of $1/\sqrt{4n}$ for any number of (n) of intervals, so there is slow convergence with increasing n. If the area of the end interval is multiplied by $\sqrt{2}$, the residual error becomes proportional to $1/n^{3/2}$, tripling the rate of convergence. The ordinary error from curvature is proportional to $1/n^2$. The latter converges more rapidly so the former may be the dominant component of residual error.

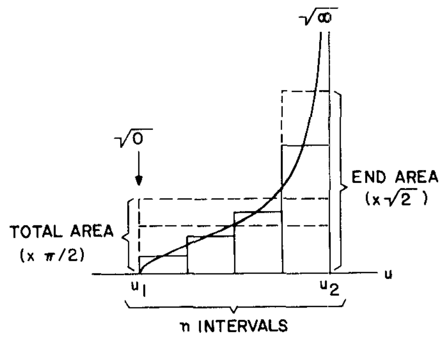

Fig. 10. Numerical integration near one bound which is a half-pole.

A further refinement at the half-pole is to take the 2/3 point instead of the midpoint, this being the centroid of the area in the interval. For this point, the multiplying factor becomes $\sqrt{4/3}$.

In either case, a still further refinement is provided by a slightly larger factor (1/0.868 for the 2/3 point).

The opposite discontinuity (a half-zero $\sqrt{0}$) may appear at the other bound, as in Fig. 10. The midpoint area is slightly too great, so it may be multiplied by $\sqrt{8/9}$ (or 0.94) as a first-order correction.

If the width of the thickness area (u_2-u_1) is much less than the other intervals ($2u_1, 1-u_2$), there is a simple rule based on one midpoint, as indicated in Fig. 10. The average is simply $\pi/2$ times the midpoint.

A variety of examples have been computed by numerical integration according to Fig. 9. Each example may be based on a specified value of w'/h, for which R_1 can be computed exactly by Appendix A or approximately by (4). This value of w'/h determines u_2. The thickness ratio t/h would then determine u_1, but there is no close explicit formula. Instead, a target value t'/h may be specified, from which an approximate value of u_1 can be computed by this formula:

$$u_1=u_2-\frac{1-u_2^2}{h/t'+1/2-u_2}. \quad (55)$$

The most significant ratio from this exercise is $\Delta w/t$, which is of the order of unity and is weakly dependent on the thickness ratio (t/h) and the width ratio (w/h or w'/h). It is a small difference so its close approximation is an achievement.

The integration of each area may be performed with an interval 1/8 the width of the thickness area shown in Fig. 10. This interval leaves a relative error < 0.01 of $\Delta w/t$

for the widest cases and much less for most cases. Its occurrence and sense are such that it does not cause an excess over the error tolerance stated for (12), (13) in comparison therewith.

The usefulness of numerical integration decreases with greater width because u_1 and u_2 approach the pole at $u=1$. This is found to decrease the rate of convergence with smaller intervals.

Acknowledgment

This study has been stimulated by the need for more knowledge of the subject line, which is used where the greatest degree of refinement is afforded. Especially it has been responsive to the opportunity afforded by the "close support" available from such small calculators as the HP-25 and the HP-97. They offer an alternative which has some practical advantages over reading a graph, though they are most valuable for developing the formulas needed in any case. In reference to this and the preceding article, the author is especially grateful to Gerri Fargnoli for the careful typing and to Walter Anderson for his skillful preparation of the graphs.

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Characteristic Impedance of a Rectangular Coaxial Line with Offset Inner Conductor

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***Abstract*—The singular-integral-equation technique is used to derive the capacitance and, hence, characteristic impedance of a rectangular coaxial line with a zero-thickness inner conductor. The position of the inner conductor is arbitrary, but its orientation is assumed to be parallel to the top and bottom walls of the outer conductor. Simple yet very accurate formulas for the capacitance and characteristic impedance are found in terms of complete elliptic integrals.**

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I. Introduction

The cross section of the rectangular transmission line analyzed in this paper is shown in Fig. 1. The zero-thickness inner conductor is arbitrarily situated but is parallel to the x axis. Both conductors are perfectly conducting, and the medium between the two conductors is a homogeneous dielectric.

This type of transmission line has found use in some EMI measurement systems [1] as a transducer for coupling EM energy from the equipment under test (EUT) into the TEM mode of the transmission line. The EUT is usually located between the inner and outer conductors

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